Section 15.2

Double Integrals Over More General Regions

PreLecture Video

1 Introduction

Double Integrals Over Arbitrary Regions

For a rectangular region $\mathcal{R} = [a, b] \times [c, d]$, the double integral

$$\iint_R f(x, y)\, dA$$

can be calculated as an iterated integral

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \qquad \text{or} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy$$

where the inner integral represents a slice of $\mathcal{R}$ at a fixed value of x or y.

In general, suppose $\mathcal{D}$ is an **arbitrary** region in $\mathbb{R}^2$. How do we calculate

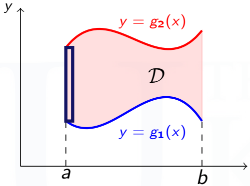
$$\iint_{\mathcal{D}} f(x, y)\, dA$$

as an iterated integral?

2 Integrating Over Horizontally and Vertically Simple Regions

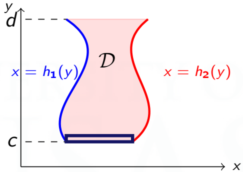
Simple Regions

Idea: When possible, slice $\mathcal{D}$ into vertical or horizontal strips.



Vertically simple region

(The upper and the lower bounds are elementary functions)



Horizontally simple region

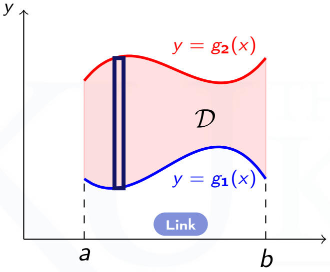
(The right and the left bounds are elementary functions)

- In these cases, we can express $\iint_{\mathcal{D}} f(x,y)\,dA$ as an iterated integral.
- The inner limits of integration are not constant, but depend on the outside variable:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \qquad\qquad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Simple Regions

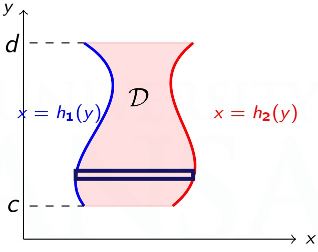
Vertically simple regions



Can be sliced into vertical strips each with constant x-coordinate

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Horizontally simple regions



Can be sliced into horizontal strips each with constant y-coordinate

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Inner limits are functions of outer variable

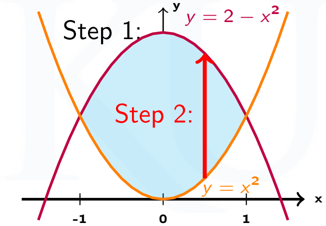
Outer limits are constants

Steps for Integration Set up

1. Draw the region of integration in $\mathbb{R}^2$.
2. Draw a horizontal arrow in the increasing x-direction if the region is horizontally simple; draw a vertical arrow in the increasing y-direction if the region is vertically simple
3. Choose the lower and upper curves using the arrow in Item 2. Solve for x in terms of y if the region is horizontally simple and for y in terms of x if the region is vertically simple.
4. Find the two constant bounds for y if horizontally simple; find the two constant bounds for x if vertically simple.
5. Set up the inner integral using bounds in Item 3 and the outer bounds using Item 4.
6. Integrate the inner first considering the outer variable constant. Replace inner bounds and integrate the outer.

Integrating over Simple Regions

Example 1: Evaluate $\iint_{\mathcal{D}} (x+2y)\,dA$ where $\mathcal{D}$ is the region between $y - x^2 = 0$ and $y + x^2 = 2$.



Solution: The first step is to draw $\mathcal{D}$. The curves intersect at $(-1,1)$ and $(1,1)$, so

$$\mathcal{D} = \{(x,y) : \underbrace{x^2 \le y \le 2-x^2}_{\text{Step 3}},\ \underbrace{-1 \le x \le 1}_{\text{Step 4}}\}.$$

$$\iint_{\mathcal{D}} (x+2y)\,dA = \underbrace{\int_{x=-1}^{x=1} \left(\int_{y=x^2}^{y=2-x^2} (x+2y)\,dy \right) dx}_{\text{Step 5}}$$

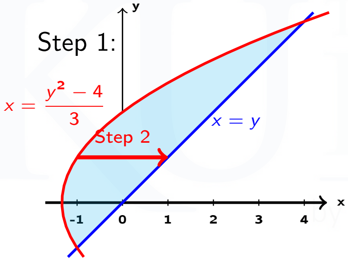
$$= \int_{x=-1}^{x=1} (xy+y^2)\Big|_{y=x^2}^{y=2-x^2} dx = \int_{-1}^{1} x(2-x^2) + (2-x^2)^2 - (x(x^2)+(x^2)^2)\,dx$$

$$= \int_{-1}^{1} (4+2x-4x^2-2x^3)\,dx = \frac{16}{3}$$

Integrating over Simple Regions

Example 2: Evaluate $\iint_{\mathcal{D}} xy\, dA$ where $\mathcal{D}$ is the region bounded by $y = x$ and $y^2 = 3x + 4$.

Step 1:



Solution: Start by drawing $\mathcal{D}$ and observe that it is horizontally simple. For each boundary curve, express x as a function of y:

$$x = y \qquad x = \frac{y^2 - 4}{3}$$

Intersection points: $(-1, -1)$ and $(4, 4)$.

So

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid \underbrace{\frac{y^2 - 4}{3} \le x \le y}_{\text{Step 3}},\ \underbrace{-1 \le y \le 4}_{\text{Step 4}} \right\}$$

Integrating over Simple Regions

Example 2 (continued):

$$\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 \mid -1 \le y \le 4,\ \frac{y^2-4}{3} \le x \le y\right\}$$

$$\iint_{\mathcal{D}} xy\, dA = \underbrace{\int_{y=-1}^{y=4} \int_{x=\frac{y^2-4}{3}}^{x=y} xy\, dx\, dy}_{\text{Step 5}}$$

$$= \int_{y=-1}^{y=4} \left[\frac{1}{2} x^2 y \Big|_{x=\frac{y^2-4}{3}}^{x=y} \right] dy$$

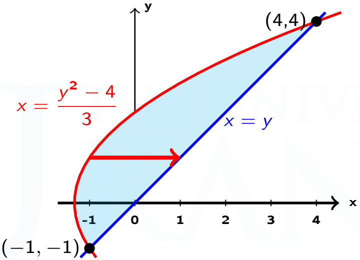
$$= \int_{y=-1}^{y=4} \frac{1}{2}(y^2)y - \frac{1}{2}\left(y \frac{\overbrace{(y^2-4)^2}^{y^4-8y^2+16}}{9} \right) dx$$

$$= \int_{-1}^{4} \left(\frac{-8y}{9} + \frac{17y^3}{18} - \frac{y^5}{18} \right) dy$$

$$= -\frac{4y^2}{9} + \frac{17y^4}{72} + \frac{y^6}{108} \Big|_{-1}^{4} = \frac{125}{8}.$$

Integrating over Simple Regions

Example 2 (continued): Take another look at the region $\mathcal{D}$.

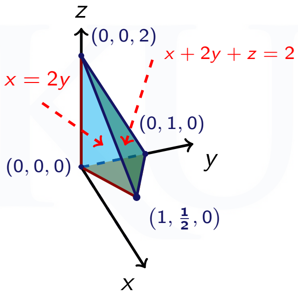


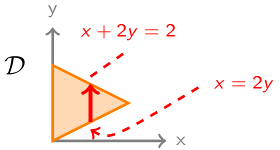
- While it is technically possible to represent $\mathcal{D}$ as a set of iterated integrals in order $\iint_{\mathcal{D}} xy\,dy\,dx$, the lower limit of integration will not be elementary:

$$y = \begin{cases} -\sqrt{3x+4} & \text{if } x < -1, \\ x & \text{if } x > -1 \end{cases}$$

Example 3: Find the volume of the tetrahedron bounded by the planes:

$$x + 2y + z = 2 \qquad x = 2y \qquad x = 0 \qquad z = 0$$





Link

Solution: The tetrahedron lies under $z = 2 - x - 2y$ and above the vertically simple region

$$\mathcal{D} = \left\{ (x, y) \;\middle|\; 0 \le x \le 1, \quad \frac{x}{2} \le y \le 1 - \frac{x}{2} \right\}$$

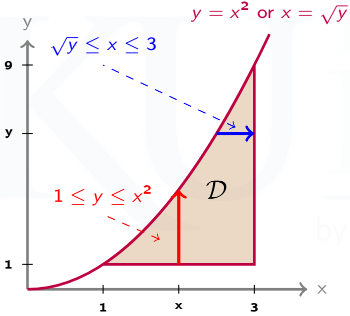
$$\begin{aligned} \text{Volume} &= \iint_{\mathcal{D}} (2 - x - 2y)\, dA \\ &= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y)\, dy\, dx \\ &= \int_0^1 (x^2 - 2x + 1)\, dx = \frac{1}{3} \end{aligned}$$

3 Reversing the Order of Integration

Changing the Order of Integration

Some regions are both vertically and horizontally simple.

Example 4:



$$\mathcal{D} = \{(x,y) \mid 1 \leq x \leq 3,\ 1 \leq y \leq x^2\}$$

(vertically simple)

$$= \{(x,y) \mid 1 \leq y \leq 9,\ \sqrt{y} \leq x \leq 3\}$$

(horizontally simple)

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_1^3 \int_1^{x^2} f(x,y)\,dy\,dx = \int_1^9 \int_{\sqrt{y}}^3 f(x,y)\,dx\,dy$$

Which iterated integral should you use? **Whichever is more convenient.**

Reversing the Order Steps

The main reason for changing the order of integration is that the given order results in a non-elementary antiderivative; in those cases, changing the order may be helpful.

1. Use the given (current) order and draw a region with the arrow corresponding to the given order.
2. Change the arrow form horizontal to vertical or vice versa to reverse the order. Follow Steps 3-6 of integration set-up.

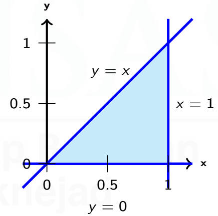
Reversing the Order of Integration

Example 5: Evaluate $\displaystyle\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

Since $\displaystyle\int e^{x^2}\, dx$ cannot be evaluated, we need to do something new.

Solution: First, draw the domain of integration:

$$\mathcal{D} = \{(x, y) \mid 0 \le y \le 1,\ y \le x \le 1\}$$



It is doubly simple, so it can also be expressed as

$$\mathcal{D} = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$$

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy \;=\; \iint_{\mathcal{D}} e^{x^2}\, dA \;=\; \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

Changing the Order of Integration

Example 5 (continued):

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \iint_{\mathcal{D}} e^{x^2}\,dA = \int_0^1 \int_0^x e^{x^2}\,dy\,dx$$

$$= \int_0^1 \left[e^{x^2} y \Big|_{y=0}^{y=x} \right] dx$$

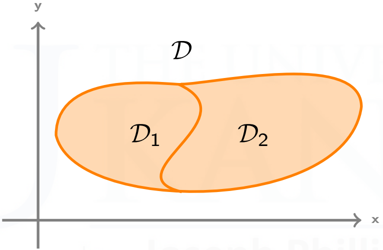
$$= \int_0^1 x e^{x^2}\,dx$$

$$= \frac{e^{x^2}}{2}\Bigg|_{x=0}^{x=1} = \frac{1}{2}(e-1).$$

4 Integrating Over More General Regions

Integrating Over General Regions

The region of integration can be subdivided:

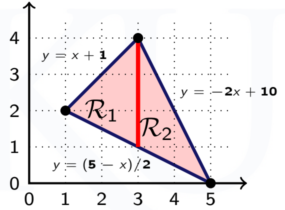


If $\mathcal{D}$ is the union of $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ don't overlap except on their boundaries, then

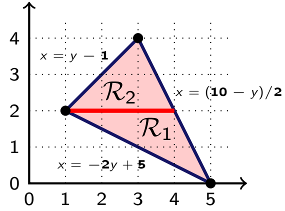
$$\iint_{\mathcal{D}} f(x,y)\,dA = \iint_{\mathcal{D}_1} f(x,y)\,dA + \iint_{\mathcal{D}_2} f(x,y)\,dA$$

Subdividing Regions

Example 6: The triangle $\mathcal{D}$ with vertices $(1,2)$, $(5,0)$, and $(3,4)$ can be subdivided two different ways, giving two different iterated integrals for $\iint_{\mathcal{D}} f\, dA$.



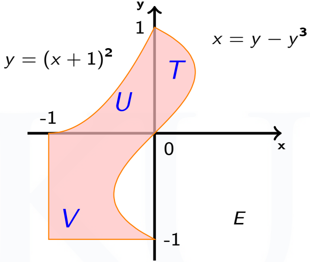
$$\underbrace{\int_1^3 \int_{(5-x)/2}^{x+1} f\, dy\, dx}_{\mathcal{R}_1} + \underbrace{\int_3^5 \int_{(5-x)/2}^{-2x+10} f\, dy\, dx}_{\mathcal{R}_2}$$



$$\underbrace{\int_0^2 \int_{-2y+5}^{(10-y)/2} f\, dx\, dy}_{\mathcal{R}_1} + \underbrace{\int_2^4 \int_{y-1}^{(10-y)/2} f\, dx\, dy}_{\mathcal{R}_2}$$

Subdividing Regions (optional)

Example 7:



Region E can be decomposed into 3 simple regions T, U, V:

T :	$0 \le y \le 1$	$0 \le x \le y - y^3$
U :	$-1 \le x \le 0$	$0 \le y \le (x+1)^2$
V :	$-1 \le y \le 0$	$-1 \le x \le y - y^3$

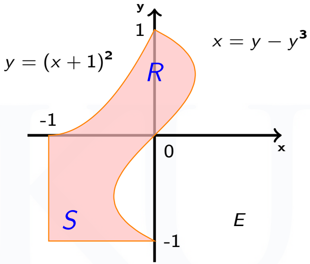
Note that T, V are horizontally simple and U is vertically simple as shown.

$$\iint_E f\,dA = \iint_T f\,dA + \iint_U f\,dA + \iint_V f\,dA$$

$$= \underbrace{\int_0^1 \int_0^{y-y^3} f\,dx\,dy}_{T} + \underbrace{\int_{-1}^0 \int_0^{(x+1)^2} f\,dy\,dx}_{U} + \underbrace{\int_{-1}^0 \int_{-1}^{y-y^3} f\,dx\,dy}_{V}$$

Subdividing Regions

Example 7 (continued):



Solving for y in $x = y - y^3$ is not possible but solving for x in $y = (x+1)^2$ is possible. Now, Region E can be decomposed into 2 simple Regions R, S:

$$\begin{array}{llrl} R: & 0 \le y \le 1 & \sqrt{y} - 1 & \le x \le y - y^3 \\ S: & -1 \le y \le 0 & -1 & \le x \le y - y^3 \end{array}$$

Note that both R, S are horizontally simple.

$$\iint_E f\,dA = \iint_R f\,dA + \iint_S f\,dA$$

$$= \underbrace{\int_0^1 \int_{\sqrt{y}-1}^{y-y^3} f\,dx\,dy}_{R} + \underbrace{\int_{-1}^0 \int_{-1}^{y-y^3} f\,dx\,dy}_{S}$$

5 Properties of Double Integrals

by Joseph Phillip Brennan
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Facts about Double Integrals

- If f and g are functions on $\mathcal{D}$, then

$$\iint_{\mathcal{D}} (f(x,y) + g(x,y))\, dA = \iint_{\mathcal{D}} f(x,y)\, dA + \iint_{\mathcal{D}} g(x,y)\, dA.$$

- If k is a constant, then $\iint_{\mathcal{D}} kf(x,y)\, dA = k \iint_{\mathcal{D}} f(x,y)\, dA.$

- If $f(x,y) \leq g(x,y)$ for all (x,y) in $\mathcal{D}$, then

$$\iint_{\mathcal{D}} f(x,y)\, dA \ \leq \ \iint_{\mathcal{D}} g(x,y)\, dA.$$

  In this case, the volume of the solid between the graphs of f and g is

$$\iint_{\mathcal{D}} (g(x,y) - f(x,y))\, dA.$$

Facts about Double Integrals

- The area of $\mathcal{D}$ is $\displaystyle\iint_{\mathcal{D}} 1\,dA$ *(The volume of the solid under the plane $z = 1$ over $\mathcal{D}$ is the area of $\mathcal{D}$ times 1)*.

- The average value of $f(x, y)$ on $\mathcal{D}$ is

$$\overline{f} \;=\; \frac{\displaystyle\iint_{\mathcal{D}} f(x, y)\,dA}{\text{Area}(\mathcal{D})} \;=\; \frac{\displaystyle\iint_{\mathcal{D}} f(x, y)\,dA}{\displaystyle\iint_{\mathcal{D}} 1\,dA}.$$

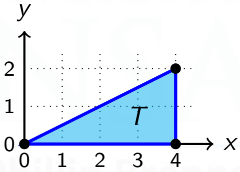
- If m, M are constants and $m \leq f(x, y) \leq M$ for all (x, y) in $\mathcal{D}$, then

$$m\,(\text{Area}(\mathcal{D})) \leq \iint_{\mathcal{D}} f(x, y)\,dA \leq M\,(\text{Area}(\mathcal{D}))\,.$$

Double Integrals: Applications

Example 8: What is the average value of $f(x, y) = \sqrt{x^2 + 9}$ on the triangle T with vertices $(0, 0)$, $(4, 0)$, and $(4, 2)$?

Solution: First, draw the triangle.



The average value is

$$\overline{f} = \frac{1}{\text{Area}(T)} \iint_T f \, dA$$

$$= \frac{1}{4} \int_0^4 \int_0^{x/2} \sqrt{x^2 + 9} \, dy \, dx = \frac{1}{4} \int_0^2 \int_{2y}^4 \sqrt{x^2 + 9} \, dx \, dy$$

The second iterated integral looks hard, so try the first one.

Double Integrals: Applications

Example 8 (continued):

$$\overline{f} = \frac{1}{4}\int_0^4 \int_0^{x/2} \sqrt{x^2+9}\, dy\, dx$$

$$= \frac{1}{4}\int_0^4 \left[y\sqrt{x^2+9} \Big|_{y=0}^{y=x/2} \right] dx$$

$$= \frac{1}{8}\int_0^4 x\sqrt{x^2+9}\, dx \qquad \text{(substitute } u = x^2+9,\ du = 2x\,dx\text{)}$$

$$= \frac{1}{16}\int_9^{25} u^{1/2}\, du \;=\; \frac{1}{24} u^{3/2}\Big|_9^{25} du \;=\; \frac{49}{12}.$$